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Integrable hierarchy for multidimensional Toda equations and topological–anti-topological fusion

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Abstract

The negative symmetry flows are incorporated into the Riemann–Hilbert problem for the homogeneous A_m-hierarchy and its $\widehat{\mathrm{gl}}(m+1,\mathbb{C})$ extension.

A loop group automorphism of order two is used to define a sub-hierarchy of $\widehat{\mathrm{gl}}(m+1,\mathbb{C})$ hierarchy containing only the odd symmetry flows. The positive and negative flows of the ± 1 grade coincide with equations of the multidimensional Toda model and of topological–anti-topological fusion.

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1. Introduction

The Riemann–Hilbert problem has a long history of applications within the theory of soliton equations (see e.g. [1,2]). Here, we make use of the Riemann–Hilbert problem to formulate multi-time evolution equations of a class of integrable models associated to $\widehat{\mathrm{gl}}(m+1)$ loop algebras. The starting point is factorization of a loop group element $G(\lambda)$:

$$G(\lambda) = G_-(\lambda)G_+(\lambda), \tag{1.1}$$

where $G_+(\lambda)$ ($G_-(\lambda)$) belong to subgroups constructed from positive (strictly negative) $\hat{\mathcal{G}}_+$ ($\hat{\mathcal{G}}_-$) graded subalgebras. The gradation is defined by powers of the spectral parameter λ counted by the grading operator $d = \lambda(\mathrm{d}/\mathrm{d}\lambda)$. Such a gradation is known as homogeneous gradation. The loop algebra $\hat{\mathcal{G}}$ decomposes into graded subspaces $\hat{\mathcal{G}} = \oplus_{n\in\mathbb{Z}}\hat{\mathcal{G}}_n$ with $\hat{\mathcal{G}}_n$

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such that $[\mathrm{d}, \hat{\mathcal{G}}_n] = n\hat{\mathcal{G}}_n$. The parameter λ plays a two-fold role; it appears as a spectral parameter in the fundamental linear spectral problem of the integrable model and also serves as a loop variable parameterizing the closed contour on the complex plane taken here to be a unit circle S^1.

The matrices appearing in the Birkhoff factorization (1.1) are linked to another important concept in the soliton theory, namely the dressing transformation which maps a vacuum to soliton solutions. In the context of the generalized Drinfeld–Sokolov formalism the dressing transformation introduces a multi-time structure associated to the positive grade Heisenberg subalgebra in $\hat{\mathcal{G}}$ [3–5]. As we vary the positive grade, the isospectral times corresponding to the generators of the Heisenberg subalgebra form an hierarchy of equations of motion. This provides a standard algebraic derivation of the integrable hierarchy which can be embedded within the Riemann–Hilbert problem. The combination of these two basic concepts allows us to derive all fundamental objects like the Hamiltonian densities and the tau function using the flows and algebraic structure inherited from the Riemann–Hilbert problem.

The well-known $\mathcal{G} = \mathrm{sl}(2)$ examples of the dressing method are mKdV and AKNS hierarchies associated to the principal and homogeneous gradations, respectively. Also, corresponding to $\mathcal{G} = \mathrm{sl}(2)$ with the principal and homogeneous gradations are the sine-Gordon and complex sine-Gordon hierarchies. However, they fall outside the scope of the standard dressing technique because their times are associated to the negative grade generators of the Heisenberg subalgebra. This motivates construction of a formalism which would incorporate both positive and negative times. The Riemann–Hilbert problem naturally allows for such a generalization [6,7]. The outcome of this construction is a unified framework with hierarchies of evolution equations corresponding to mutually commuting positive and negative flows.

In this paper, we generalize the multi-time formulation of the Riemann–Hilbert problem by including all self-commuting diagonal generators of the $\widehat{\mathrm{gl}}(m+1)$ loop algebra of positive and negative grades. It is known that in the homogeneous gradation the Heisenberg subalgebra generated by $E^{(k)} = \mu_m \cdot H^{(k)}$ (μ_m is the mth fundamental weight) with $k \in \mathbb{Z}$ has a centralizer given by $\widehat{\mathrm{gl}}(m) \times \hat{u}(1)$ which is non-Abelian for $m > 1$ (see e.g. [8]). The symmetry structure in question is given by the self-commuting symmetry flows associated not to the full centralizer of the Heisenberg subalgebra but only to the Abelian generators $E_{jj}^{(k)} = \lambda^k E_{jj}$, $j = 1, \ldots, m+1$ within it. Here, we use a notation $(E_{rs})_{ij} = \delta_{ir}\delta_{js}$. Those flows which correspond to the Heisenberg subalgebra generators $E^{(k)} = \mu_m \cdot H^{(k)}$ with positive grade ($k \in \mathbb{N}$), agree with times of the Hamiltonian evolution equations for the constrained KP hierarchy [9–13].

There are two different reasons for commutativity of flows. The positive (negative) flows commute among themselves since the commutators of their associated generators vanish. For mixed case, a straightforward calculation shows that the positive flows commute with negative flows solely as a result of their definitions.

The presence of both positive and negative sectors of the extended hierarchy agrees with a complex structure of $\widehat{\mathrm{gl}}(m+1, \mathbb{C})$ symmetry with flows generated by $E_{jj}^{(k)}$, $j = 1, \ldots, m+1$. When we consider the positive flows only, the above structure reduces naturally to the homogeneous $A_m = \widehat{\mathrm{sl}}(m+1)$-hierarchy which generalizes the AKNS hierarchy for $m > 1$.

The diagonal generators of $\widehat{\mathrm{gl}}(m+1)$ generate the multi-dimensions of the Toda model as flows with ± 1 gradations of the underlying hierarchy. Those flows take a form of the Cecotti–Vafa equations of the topological–anti-topological fusion [14,15] when considered within the sub-hierarchy restricted (or twisted) by a specific loop group automorphism. This unveils the topological field theory concepts in the context of the reduction of the homogeneous $\widehat{\mathrm{gl}}(m+1,\mathbb{C})$-hierarchy. Similar integrable structure with positive flows only was recently used to find solutions to the WDVV equations [16].

In Section 2, we define a Riemann–Hilbert problem for the integrable model with the underlying $\widehat{\mathrm{gl}}(m+1)$ loop algebra with the homogeneous gradation containing positive and negative symmetry flows. For the model with only positive multi-times this hierarchy reduces to the homogeneous A_m-hierarchy. We also study the action of associated commuting symmetry flows on the dressing matrices of positive and negative gradation and derive the conservation laws and expressions for the Hamiltonian densities. The underlying tau function is given by taking an expectation value of the Riemann–Hilbert equation based on the highest weight vacuum of the associated Kac–Moody algebra [17]. In Section 3, we discuss the positive dressing matrix M and its inverse using the relation between the algebraic and pseudo-differential approaches.

In Section 4, we derive the multidimensional Toda model equations from the positive and negative flows of ± 1 grade of the $\widehat{\mathrm{gl}}(m+1,\mathbb{C})$-hierarchy. Next, we impose the set of constraints on the dressing matrices defining a consistent sub-hierarchy allowing only odd positive and negative flows. The dressing matrices are constrained to be the fixed points of a specific loop group automorphism of order 2. The Cecotti–Vafa equations of topological–anti-topological fusion are found among the positive and negative flows of ± 1 grade of the reduced integrable sub-hierarchy. The similar sub-hierarchy (without the negative flows) has recently been shown to provide solutions of the Darboux–Egoroff system of PDEs [16]. Here, due to the presence of negative and positive flows, we obtain two coupled Darboux–Egoroff systems embedded in the complex-like structure of the Cecotti–Vafa equations. As an example we discuss the extended AKNS/complex sine-Gordon model and its reduction.

2. Extended Riemann–Hilbert problem and $\widehat{\mathbf{gl}}$ (m +1) symmetry flows

2.1. *The Riemann–Hilbert factorization for the positive flows*

We will introduce the Riemann–Hilbert problem in terms two subgroups of the Lie loop group G defined as

$$G_- = \left\{ g \in G | g(\lambda) = 1 + \sum_{i<0} g^{(i)} \right\}, \tag{2.1}$$

$$G_+ = \left\{ g \in G | g(\lambda) = \sum_{i\geq 0} g^{(i)} \right\}, \tag{2.2}$$

where g_i has grading i with respect to a homogeneous gradation defined by derivation $d = \lambda(\mathrm{d}/\mathrm{d}\lambda)$. It also holds that $G_+ \cap G_- = I$. Let the loop algebra corresponding to G be $\hat{\mathcal{G}} = \widehat{\mathrm{gl}}(m+1)$. This algebra splits into the direct sum $\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$, where $\hat{\mathcal{G}}_\pm$ are Lie algebras associated with the subgroups $G_\pm$.

We now define a Riemann–Hilbert factorization problem for the homogeneous gradation:

$$\exp\left(\sum_{j=1}^{m+1}\sum_{n=1}^{\infty} E_{jj}^{(n)} u_j^{(n)}\right) g = \Theta^{-1}(\mathbf{u},\lambda) M(\mathbf{u},\lambda), \tag{2.3}$$

where g is a constant element in G_-G_+ while $\Theta^{-1} \in G_-$, $M \in G_+$. We use the multi-time notation with $(\mathbf{u}) = (\mathbf{u}_1, \ldots, \mathbf{u}_{m+1})$ to denote $m+1$ multi-flows $\mathbf{u}_j$. Each argument $\mathbf{u}_j$, $j = 1, \ldots, m+1$ is an abbreviated notation for the multi-flows $u_j^{(n)}$ with n between 1 and ∞. We refer to the flows $u_j^{(n)}$ with $n > 0$ as positive flows to distinguish them from the negative flows associated with the negative $n < 0$ gradations, to be defined below.

Acting with $\partial/\partial u_j^{(n)}$ on both sides of (2.3) we find

$$\Theta(u) E_{jj}^{(n)} \Theta^{-1}(u) = \Theta(u)\left(\frac{\partial}{\partial u_j^{(n)}}\Theta^{-1}(u)\right) + \left(\frac{\partial}{\partial u_j^{(n)}} M(u)\right) M^{-1}(u). \tag{2.4}$$

Note, that $\Theta((\partial/\partial u_j^{(n)})\Theta^{-1})$ is in $\mathcal{G}_-$ and $((\partial/\partial u_j^{(n)})M)M^{-1}$ is in $\mathcal{G}_+$. Hence, for the (2.4) expressions:

$$\frac{\partial}{\partial u_j^{(n)}}\Theta(\mathbf{u},\lambda) = -(\Theta E_{jj}^{(n)}\Theta^{-1})_-\Theta(\mathbf{u},\lambda), \tag{2.5}$$

$$\frac{\partial}{\partial u_j^{(n)}} M(\mathbf{u},\lambda) = (\Theta E_{jj}^{(n)}\Theta^{-1})_+ M(\mathbf{u},\lambda), \tag{2.6}$$

where $(\cdot_\pm$ denote the projections into $\hat{\mathcal{G}}_\pm$.

We now address the issue of commutativity of the flows. Applying, respectively, $\partial^2/\partial u_j^{(n)}\partial u_i^{(k)}$ and $\partial^2/\partial u_i^{(k)}\partial u_j^{(n)}$ on both sides of Eq. (2.3) produces identical results due to commutativity of $E_{jj}^{(n)}$ with $E_{ii}^{(k)}$. This ensures that

$$\frac{\partial^2\Theta(\mathbf{u})}{\partial u_j^{(n)}\partial u_i^{(k)}} = \frac{\partial^2\Theta(\mathbf{u})}{\partial u_i^{(k)}\partial u_j^{(n)}}, \qquad \frac{\partial^2 M(\mathbf{u})}{\partial u_j^{(n)}\partial u_i^{(k)}} = \frac{\partial^2 M(\mathbf{u})}{\partial u_i^{(k)}\partial u_j^{(n)}}. \tag{2.7}$$

From Eq. (2.5) we find the tracelessness condition

$$\sum_{j=1}^{m+1}\frac{\partial}{\partial u_j^{(n)}}\Theta(\mathbf{u},\lambda) = 0, \tag{2.8}$$

which allows to consider the positive flows together with the negative dressing matrix $\Theta(\mathbf{u},\lambda)$ as a homogeneous A_m-hierarchy with a $\widehat{\mathrm{sl}}(m+1)$ symmetry of flows, to be discussed in the next subsection. For $m = 1$ we recover in this way the AKNS hierarchy.

2.2. The homogeneous A_m-hierarchy, isospectral flows, Hamiltonians

The flows defining conservation laws of the homogeneous A_m-hierarchy are defined in terms of $u^{(k)}_{m+1}$-flows through

$$\frac{\partial}{\partial t_k} = \frac{\partial}{\partial u^{(k)}_{m+1}}. \tag{2.9}$$

The standard dressing construction [8], associates isospectral flows to a semisimple grade-one element $E^{(1)}$:

$$E^{(1)} = \lambda E = \mu_m \cdot H^{(1)} = \frac{\lambda}{m+1} I - \lambda E_{m+1m+1}, \tag{2.10}$$

where μ_m is the mth fundamental weight of $\mathrm{sl}(m+1)$. The kernel of ad E is

$$\mathcal{K} = \mathrm{Ker}(\mathrm{ad}\, E) = \{\widehat{\mathrm{gl}}(m) \oplus \hat{u}(1)\}, \tag{2.11}$$

with the $\hat{u}(1)$ generated center $\mathcal{C}(\mathcal{K}) = \{\mu_m \cdot H^{(k)}, k \in \mathbb{Z}\}$ of $\mathrm{Ker}(\mathrm{ad}\, E)$. Each isospectral flow t_k is assigned to an element:

$$E^{(k)} \equiv \mu_m \cdot H^{(k)}, \quad k \geq 1, \tag{2.12}$$

in the center $\mathcal{C}(\mathcal{K})$ according to

$$\frac{\partial}{\partial t_k} \Theta = (\Theta E^{(k)} \Theta^{-1})_- \Theta, \tag{2.13}$$

which coincides with (2.5) for $j = m+1$ in agreement with the definition (2.9).

For $k = 1$, we obtain from (2.5)

$$\left(\frac{\partial}{\partial u^{(1)}_{m+1}} - E^{(1)}_{m+1\,m+1} - [\theta^{(-1)}, E^{(1)}_{m+1\,m+1}] \right) \Theta = -\Theta E^{(1)}_{m+1\,m+1}, \tag{2.14}$$

where $\Theta^{(-1)}$ is a term of expansion of $\Theta = 1 + \theta^{(-1)} + \mathrm{O}(\lambda^{-2})$. Eq. (2.14) can also be rewritten as

$$\left(\frac{\partial}{\partial t_1} + E^{(1)} + [\theta^{(-1)}, E^{(1)}] \right) \Theta = \Theta E^{(1)}, \tag{2.15}$$

in agreement with (2.13).

The potential $A \equiv [\theta^{(-1)}, E^{(1)}]$ lies in the grade-zero component of the image of $\mathrm{ad}(E)$ and can therefore be parameterized as

$$A = [\theta^{(-1)}, E^{(1)}] = -[\theta^{(-1)}, E^{(1)}_{m+1\,m+1}] = \sum_{i=1}^{m} (-\Psi_i E_{im+1} + \Phi_i E_{m+1i}). \tag{2.16}$$

We will refer to the hierarchy defined by the isospectral times from (2.13) and symmetry flows from Eq. (2.5) with the parameterization in (2.16) as the homogeneous A_m-hierarchy [19–21].

Given the parameterization (2.16), Eq. (2.14) can be cast into the form

$$\Theta^{-1} L \Theta = \partial_x - E^{(1)}_{m+1\, m+1}, \tag{2.17}$$

involving the matrix Lax operator L:

$$L = \partial_x - E^{(1)}_{m+1\, m+1} + A, \tag{2.18}$$

where $\partial_x = \partial/\partial t_1 = \partial/\partial u^{(1)}_{m+1}$ is acting to the right as an operator according to the Leibniz rule. In [22], the homogeneous A_m hierarchy was constructed by rotating matrices of the Lax operator (2.18) into the kernel of $\mathrm{ad}(E^{(1)})$.

The linear spectral problem emerges when setting $n = 1$ and $j = m + 1$ in (2.6). This reveals the matrix M as a solution of [7]:

$$L(M) = (\partial_x - E^{(1)}_{m+1\, m+1} + A)(M) = 0. \tag{2.19}$$

We will now discuss the conservation laws of the homogeneous A_m-hierarchy. These laws are based on requirement of locality with respect to the potentials Φ_i, Ψ_i from (2.16). A quantity is considered local if it is a polynomial of Φ_i, Ψ_i and their derivatives.

From Eq. (2.5) we find

$$\frac{\partial}{\partial u^{(n)}_i} (\Theta E^{(n')}_{jj} \Theta^{-1}) = [\Theta E^{(n')}_{jj} \Theta^{-1}, (\Theta E^{(n)}_{ii} \Theta^{-1})_-], \tag{2.20}$$

and therefore

$$\begin{aligned} &\frac{\partial}{\partial u^{(n)}_i} \mathrm{Tr}_0 (E^{(1)} \Theta E^{(n')}_{jj} \Theta^{-1}) - \frac{\partial}{\partial u^{(n')}_j} \mathrm{Tr}_0 (E^{(1)} \Theta E^{(n)}_{ii} \Theta^{-1}) \\ &\quad = \mathrm{Tr}_0 (E^{(1)} [\Theta E^{(n')}_{jj} \Theta^{-1}, \Theta E^{(n)}_{ii} \Theta^{-1}]) = 0, \end{aligned} \tag{2.21}$$

with the trace which includes projection on the λ^0 term:

$$\mathrm{Tr}_0(XY) \equiv \langle X, Y \rangle_0 = \sum_{i+j=0} \mathrm{tr}(X_i Y_j), \quad X = \sum_i X_i \lambda^i, \quad Y = \sum_i Y_i \lambda^i. \tag{2.22}$$

It is therefore natural to associate to the Riemann–Hilbert factorization approach the quantities:

$$\mathcal{H}^{(n)}_j = \mathrm{Tr}_0 (E^{(1)} \Theta E^{(n)}_{jj} \Theta^{-1}), \quad j = 1, \ldots, m+1, \tag{2.23}$$

which satisfy according to (2.21) the identity

$$\frac{\partial \mathcal{H}^{(n)}_j}{\partial u^{(n')}_k} - \frac{\partial \mathcal{H}^{(n')}_k}{\partial u^{(n)}_j} = 0, \quad n, n' > 0. \tag{2.24}$$

Furthermore, by a simple differentiation one can verify that $\mathcal{H}^{(n)}_j$ is a derivative $\mathcal{H}^{(n)}_j = \partial_x \mathcal{J}^{(n)}_j$ of a current density:

$$\begin{aligned} &\mathcal{J}^{(n)}_j = -\mathrm{Tr}_0 \left(\lambda \frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} E^{(n)}_{jj} \Theta^{-1} \right) = -\mathrm{Res}_\lambda \left(\mathrm{tr} \left(E^{(n)}_{jj} \Theta^{-1} \frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} \right) \right), \\ &j = 1, \ldots, m+1. \end{aligned} \tag{2.25}$$

The current density $\mathcal{J}_j^{(n)}$ satisfies the identity:

$$\frac{\partial \mathcal{J}_j^{(n)}}{\partial u_k^{(n')}} - \frac{\partial \mathcal{J}_k^{(n')}}{\partial u_j^{(n)}} = 0, \quad n, n' > 0, \tag{2.26}$$

analogous to (2.24).

The quantity

$$\frac{\partial \mathcal{J}_j^{(n)}}{\partial u_k^{(n')}} = \mathrm{Tr}_0\left(\lambda \frac{\mathrm{d}(\Theta E_{kk}^{(n')}\Theta^{-1})_+}{\mathrm{d}\lambda}\Theta E_{jj}^{(n)}\Theta^{-1}\right) \tag{2.27}$$

becomes local (in terms of Ψ_i, Φ_i) for $j = k = m+1$, as observed in [22]. We therefore obtain the local conservation laws in terms of

$$\mathcal{H}_n = \mathcal{H}_{m+1}^{(n)} = -\mathrm{Tr}_0(E^{(1)}\Theta E^{(n)}\Theta^{-1}), \tag{2.28}$$

and

$$\mathcal{J}_n = \mathcal{J}_{m+1}^{(n)} = \mathrm{Tr}_0\left(\lambda \frac{\mathrm{d}\Theta}{\mathrm{d}\lambda} E^{(n)}\Theta^{-1}\right) = \mathrm{Res}_\lambda\left(\mathrm{tr}\left(E^{(n)}\Theta^{-1}\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda}\right)\right), \tag{2.29}$$

connected via $\mathcal{H}_n = \partial_x \mathcal{J}_n$. The observation that (2.27) becomes local for $j = k = m+1$ provides a direct way to prove conservation of the Hamiltonians $H_n = \int \mathcal{H}_n \,\mathrm{d}x$. One notices that

$$\frac{\partial}{\partial t_{n'}} H_n = \int \partial_x \frac{\partial}{\partial t_{n'}} \mathcal{J}_n = 0, \quad n, n' > 0, \tag{2.30}$$

due to locality of the relevant expression in (2.27) for $j = k = m+1$.

Consider the following expansions:

$$\Theta = 1 + \sum_{k=1}^{\infty} \frac{\Theta^{(-k)}}{\lambda^k}, \tag{2.31}$$

$$\Theta E^{(1)}\Theta^{-1} = E^{(1)} + A + \sum_{k=1}^{\infty}\frac{A^{(-k)}}{\lambda^k}, \tag{2.32}$$

$$\Theta E^{(n)}\Theta^{-1} = \lambda^n E + \lambda^{n-1}A + \sum_{k=1}^{\infty}\lambda^{n-k-1}A^{(-k)}. \tag{2.33}$$

Plugging $j = k = m+1$ and $n' = 1$ into the identity (2.26) yields

$$\mathrm{tr}(EA^{(-n)}) = -\frac{1}{2}\sum_{k=0}^{n-1}\mathrm{tr}(A^{(-k)}A^{(1+k-n)}), \quad n \geq 1. \tag{2.34}$$

In terms of the elements of expansions (2.31)–(2.33) $\mathcal{H}_n$ takes the form

$$\mathcal{H}_n = -\mathrm{tr}(EA^{(-n)}) = -\frac{1}{2}\sum_{k=0}^{n-1}\mathrm{tr}(A^{(-k)}A^{(1+k-n)}), \quad n \geq 1, \tag{2.35}$$

which allow us to cast Hamiltonians into the well-known form due to [20]

$$\mathcal{H}_n = \tfrac{1}{2}\mathrm{Tr}_0(\lambda^{n+1} X^2(\lambda)) \quad \text{for} \quad n \geq 1, \tag{2.36}$$

where

$$X(\lambda) = \sum_{i=1}^{\infty} X_i \lambda^{-i} = \sum_{k=1}^{\infty} A^{(-k+1)} \lambda^{-k} = (\Theta E^{(0)} \Theta^{-1})_-, \tag{2.37}$$

where $A^{(0)}$ denotes the potential A. $\mathcal{H}_n$ can also be written in an equivalent form

$$\mathcal{H}_n = \frac{1}{2} \langle \lambda^n X, X \rangle_{-1} = \frac{1}{2} \oint \mathrm{d}\lambda\, \lambda^n\, \mathrm{tr}(X^2), \tag{2.38}$$

where we introduced another symmetric bilinear form

$$\langle X, Y \rangle_{-1} \equiv \mathrm{Res}_\lambda(\mathrm{tr}(XY)) = \sum_{i+j=-1} \mathrm{tr}(X_i Y_j). \tag{2.39}$$

From

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \frac{1}{2} \langle \lambda^n (X + \epsilon Y)^2 \rangle_{-1} \bigg|_{\epsilon=0} = \langle \lambda^n X, Y \rangle_{-1} = \langle \nabla \mathcal{H}_n, Y \rangle_{-1}, \tag{2.40}$$

we identify $\nabla \mathcal{H}_n = \lambda^n X$ which according to the Adler–Kostant–Symes formalism yields for the flows:

$$\frac{\partial X(\lambda)}{\partial t_n} = [(\lambda^n X(\lambda))_-, X(\lambda)] = [(\Theta E^n \Theta^{-1})_-, X(\lambda)], \tag{2.41}$$

or equivalently

$$\frac{\partial X(\lambda)}{\partial t_n} = -[(\lambda^n X(\lambda))_+, X(\lambda)]_-, \tag{2.42}$$

which fully agrees with the dressing formula in (2.20).

2.3. Extended Riemann–Hilbert factorization problem, negative flows

We now define an extended (with positive and negative flows) Riemann–Hilbert factorization problem for the homogeneous gradation:

$$\exp\left(\sum_{j=1}^{m+1} \sum_{n=1}^{\infty} E_{jj}^{(n)} u_j^{(n)} \right) g \exp\left(- \sum_{j=1}^{m+1} \sum_{n=1}^{\infty} E_{jj}^{(-n)} u_j^{(-n)} \right) = \Theta^{-1}(\mathbf{u}, \lambda) M(\mathbf{u}, \lambda). \tag{2.43}$$

As before g is a constant element in $G_- G_+$ while $\Theta^{-1} \in G_-$, $M \in G_+$. Also, again we use the multi-time notation with $(\mathbf{u}) = (\mathbf{u}_1, \ldots, \mathbf{u}_{m+1})$ to denote $m+1$ multi-flows $\mathbf{u}_j$ but now in $\mathbf{u}_j$, $j = 1, \ldots, m+1$ the multi-flows $u_j^{(n)}$ depend on n between $-\infty$ and ∞.

The exponential term $\exp(-\sum_{j=1}^{m+1}\sum_{n=1}^{\infty} E_{jj}^{(-n)} u_j^{(-n)})$ on the left-hand side of (2.43) contains terms of negative grade. This term extends the standard Riemann–Hilbert problem of the KP like models by negative flows.

It is known, that the standard Riemann–Hilbert factorization holds for $GL(m, \mathbb{C})$the sufficiently small values of the flow parameters appearing in the exponential [6,18]. Here we write formally an infinite sum over the flow parameters in each exponential term on the left-hand side of Eq. (2.43). Since our interest in the generalized Riemann–Hilbert factorization is in deriving an hierarchy of evolution equations for the Abelian flows we can in principle truncate the formal infinite sums in the exponential functions and still obtain the information we are seeking for sufficiently many flows.

In addition, to the evolution equations (2.5) and (2.6), which still hold in the extended case, we also encounter the new hierarchy of equations governing the evolution of the negative multi-flows. After applying $\partial/\partial u_j^{(-n)}$ on both sides of (2.43) one finds

$$\frac{\partial}{\partial u_j^{(-n)}} \Theta(\mathbf{u}, \lambda) = (M E_{jj}^{(-n)} M^{-1})_- \Theta(\mathbf{u}, \lambda), \tag{2.44}$$

$$\frac{\partial}{\partial u_j^{(-n)}} M(\mathbf{u}, \lambda) = -(M E_{jj}^{(-n)} M^{-1})_+ M(\mathbf{u}, \lambda). \tag{2.45}$$

One observes that the above system of flow equations is invariant under the right multiplication of $M(\mathbf{u}, \lambda)$ by a constant diagonal matrix.

Similarly, we find the tracelessness condition:

$$\sum_{j=1}^{m+1} \frac{\partial}{\partial u_j^{(-n)}} M(\mathbf{u}, \lambda) = 0, \tag{2.46}$$

which allows to consider the negative flows together with the positive dressing matrix $M(\mathbf{u}, \lambda)$ as a homogeneous A_m-hierarchy, which generalizes the complex sine-Gordon equation hierarchy [7]. For the hierarchy containing both positive and negative flows and the dressing matrices $\Theta(\mathbf{u}, \lambda)$ and $M(\mathbf{u}, \lambda)$ we work with the full $\widehat{gl}(m+1)$ symmetry.

Let us now discuss commutativity of all flows. Applying, respectively, $\partial^2/\partial u_j^{(-n)} \partial u_i^{(-k)}$ and $\partial^2/\partial u_i^{(-k)} \partial u_j^{(-n)}$ on both sides of Eq. (2.43) produces identical results due to commutativity of $E_{jj}^{(-n)}$ with $E_{ii}^{(-k)}$. This ensures that

$$\frac{\partial^2 \Theta(\mathbf{u})}{\partial u_j^{(-n)} \partial u_i^{(-k)}} = \frac{\partial^2 \Theta(\mathbf{u})}{\partial u_i^{(-k)} \partial u_j^{(-n)}}, \qquad \frac{\partial^2 M(\mathbf{u})}{\partial u_j^{(-n)} \partial u_i^{(-k)}} = \frac{\partial^2 M(\mathbf{u})}{\partial u_i^{(-k)} \partial u_j^{(-n)}}. \tag{2.47}$$

In a mixed case we apply $\partial^2/\partial u_j^{(n)} \partial u_i^{(-k)}$ and $\partial^2/\partial u_i^{(-k)} \partial u_j^{(n)}$ on both sides of Eq. (2.43). The result is

$$\begin{aligned}\frac{\partial^2 \Theta(\mathbf{u})}{\partial u_j^{(-n)} \partial u_i^{(k)}} &= -([(M E_{jj}^{(-n)} M^{-1})_-, \Theta E_{ii}^{(k)} \Theta^{-1}])_- \Theta \\ &\quad -(\Theta E_{ii}^{(k)} \Theta^{-1})_- (M E_{jj}^{(-n)} M^{-1})_- \Theta,\end{aligned} \tag{2.48}$$

$$\frac{\partial^2 \Theta(\mathbf{u})}{\partial u_i^{(k)} \partial u_j^{(-n)}} = ([(\Theta E_{ii}^{(k)} \Theta^{-1})_+, ME_{jj}^{(-n)} M^{-1}])_- \Theta$$
$$- (ME_{jj}^{(-n)} M^{-1})_- (\Theta E_{ii}^{(k)} \Theta^{-1})_- \Theta. \tag{2.49}$$

One finds that the right-hand sides of Eqs. (2.48) and (2.49) are equal and cancel each other's contributions to the commutator:

$$\left[\frac{\partial}{\partial u_j^{(-n)}}, \frac{\partial}{\partial u_i^{(k)}}\right] \Theta = 0. \tag{2.50}$$

Note, that this time proof for the commutativity follows automatically from the construction and does not rely on the commutativity of $E_{jj}^{(n)}$ with $E_{ii}^{(-k)}$.

Hence it holds that

$$\frac{\partial^2 \Theta(\mathbf{u})}{\partial u_j^{(n)} \partial u_i^{(-k)}} = \frac{\partial^2 \Theta(\mathbf{u})}{\partial u_i^{(-k)} \partial u_j^{(n)}}, \tag{2.51}$$

and analogous arguments lead to the same identity with matrix M replacing Θ.

2.4. The tau function

The identity (2.26) suggests that $\mathcal{J}_j^{(n)}$ can be written as $\partial/\partial u_j^{(n)}$ of some function of the Φ_i, Ψ_i variables. Conventionally, this function is denoted as the logarithm of the τ-function. Accordingly, the τ-function is introduced by the relation:

$$\mathcal{J}_j^{(n)} = -\frac{\partial \log \tau}{\partial u_j^{(n)}}, \tag{2.52}$$

with more conventional identity

$$\mathcal{J}_n = -\frac{\partial}{\partial t_n} \log \tau, \tag{2.53}$$

being a special case corresponding to $j = m + 1$. Note, that the relation

$$\frac{\partial \mathcal{J}_n}{\partial t_{n'}} - \frac{\partial \mathcal{J}_{n'}}{\partial t_n} = 0, \quad n, n' > 0 \tag{2.54}$$

also holds as a special case of (2.26).

We will use below the setting of A_m Kac–Moody algebra to integrate these equations to obtain a closed expression for the τ-function.

Let the elements of the Kac–Moody algebra be $\xi + s\hat{k}$, where $\xi(\varphi) \equiv \xi^a(\varphi) T_a$ is a function on S^1 with values in the underlying finite-dimensional Lie algebra $\mathcal{G} = \mathrm{sl}(m + 1)$ and s is the central element. The Kac–Moody algebra reads:

$$[\xi_1 + s_1\hat{k}, \xi_2 + s_2\hat{k}] = [\xi_1, \xi_2] + \hat{k}\omega(\xi_1, \xi_2) = [\xi_1, \xi_2] + \hat{k} \oint \frac{\mathrm{d}\varphi}{2\pi} \mathrm{tr}(\partial_\varphi \xi_1 \xi_2),$$

where φ is an S^1 angle variable. The adjoint action of the group G on $\mathcal{G}$

$$\mathrm{Ad}(g)\xi = g\xi g^{-1},$$

generalizes to

$$\mathrm{Ad}_g(\xi + s\hat{k}) = g\xi g^{-1} + \hat{k}\left(s + \oint \frac{\mathrm{d}\varphi}{2\pi}\mathrm{tr}(g^{-1}\partial_\varphi g\xi)\right), \tag{2.55}$$

which in terms of the loop variable $\lambda = \exp(\mathrm{i}\varphi)$ takes the following form:

$$\mathrm{Ad}_g(\xi + s\hat{k}) = g\xi g^{-1} + \hat{k}\left(s + \mathrm{Res}_\lambda\,\mathrm{tr}\left(g^{-1}\frac{\mathrm{d}g}{\mathrm{d}\lambda}\xi\right)\right). \tag{2.56}$$

In particular,

$$\mathrm{Ad}_\Theta(E^{(n)}) = \Theta E^{(n)}\Theta^{-1} + \hat{k}\,\mathrm{Res}_\lambda\,\mathrm{tr}\left(\Theta^{-1}\frac{\mathrm{d}\Theta}{\mathrm{d}\lambda}E^{(n)}\right) = \Theta E^{(n)}\Theta^{-1} + \hat{k}\mathcal{J}_n. \tag{2.57}$$

Let $|0\rangle$ be the highest weight state such that $X_{\geq 0}|0\rangle = 0$ and $\langle 0|X_{\leq 0} = 0$ with $\langle 0|\hat{k}|0\rangle = 1$. Then

$$\langle 0|X_+X_-|0\rangle = \omega(X_+, X_-) = \mathrm{Res}_\lambda\,\mathrm{tr}\left(\frac{\mathrm{d}X_+}{\mathrm{d}\lambda}X_-\right). \tag{2.58}$$

The τ-function is defined by taking the centrally extended the Riemann–Hilbert formula (2.43) and multiplying from left and right by the vacuum states:

$$\tau(\mathbf{u}) = \left\langle 0\left|\exp\left(\sum_{j=1}^{m+1}\sum_{n=1}^{\infty}E_{jj}^{(n)}u_j^{(n)}\right)g\exp\left(-\sum_{j=1}^{m+1}\sum_{n=1}^{\infty}E_{jj}^{(-n)}u_j^{(-n)}\right)\right|0\right\rangle, \tag{2.59}$$

similar to the definition in [17] for the AKNS hierarchy. In the above definition the τ-function depends on all the symmetry flows of the extended hierarchy. Commutativity of all flows extends to the affine Kac–Moody case. The presence of a central element does not spoil the commutativity between positive and negative flows. The cancellation between the commutators shown in Eq. (2.50) extends to the affine case.

Alternatively, for $\hat{M}$ being in a central extension $\hat{G}$ of G over M we can write the τ-function as $\tau(\mathbf{u}) = \langle 0|\hat{M}|0\rangle = \langle 0|\hat{M}_0|0\rangle$ since the zero-grade term containing $\hat{k}$ resides in $\hat{M}_0$.

By setting all the negative flows $u_j^{(-n)}$ and positive flows $u_j^{(-n)}$, $j \neq m+1$ to zero in (2.59) we recover the standard τ-function for the homogeneous A_m-hierarchy [17]. From (2.59) we find

$$\frac{\partial}{\partial t_n}\tau(\mathbf{u}) = -\langle 0|E^{(n)}\hat{\Theta}^{-1}(\mathbf{u})\hat{M}(\mathbf{u})|0\rangle = -\langle 0|E^{(n)}\hat{\Theta}^{-1}(\mathbf{u})|0\rangle\tau(\mathbf{u}). \tag{2.60}$$

Using the property $\langle 0|X_- = 0$ of the highest weight state, the last equation can be rewritten as

$$\frac{\partial}{\partial t_n}\tau(\mathbf{u}) = -\langle 0|\mathrm{Ad}_\Theta(E^{(n)})|0\rangle\tau(\mathbf{u}) = -\mathcal{J}_n\tau(\mathbf{u}), \tag{2.61}$$

which confirms that the definition (2.59) reproduces the τ-function defined previously in (2.53). More generally we find

$$\frac{\partial}{\partial u_j^{(n)}}\tau(\mathbf{u}) = \langle 0|\mathrm{Ad}_\Theta(E_{jj}^{(n)})|0\rangle\tau(\mathbf{u}) = -\mathcal{J}_j^{(n)}\tau(\mathbf{u}). \tag{2.62}$$

3. The pseudo-differential formalism and construction of the M matrix

In this section we construct explicitly the $M(\mathbf{u}, \lambda)$ and $M^{-1}(\mathbf{u}, \lambda)$ matrices in terms of the objects appearing in the pseudo-differential Lax calculus approach to the homogeneous A_m hierarchy. The matrix M will be constructed as a expansion in positive powers of λ which solves the spectral linear problem $L(M) = 0$ in (2.19). This will be accomplished by establishing relation between the algebraic approach and the equivalent one based on the pseudo-differential Lax operator [9–13]:

$$\mathcal{L} = \partial_x + \sum_{i=1}^{m} \Phi_i \partial_x^{-1} \Psi_i, \tag{3.1}$$

and its inverse $\mathcal{L}^{-1}$. Both operators can be represented as ratios of two monic ordinary differential operators of order m and $m+1$ [7,23,24]:

$$\mathcal{L} = L_{m+1} L_m^{-1} = \partial_x + \sum_{i=1}^{m} L_{m+1}(\phi_i) \partial_x^{-1} \psi_i, \tag{3.2}$$

$$\mathcal{L}^{-1} = L_m L_{m+1}^{-1} = \sum_{j=1}^{m+1} L_m(\bar{\phi}_j) \partial_x^{-1} \bar{\psi}_j. \tag{3.3}$$

Next, we provide few definitions and lemmas regarding the ordinary differential operators and Wronskians necessary for technical proofs to be shown in this section. Let L_m be a monic differential operator of order m: $L_m = \partial_x^m + u_{m-1}\partial_x^{m-1} + \cdots + u_1\partial_x + u_0$ and let ϕ_i, $i = 1, \ldots, m$ be a basis for $\operatorname{Ker} L_m = \{\phi_1, \ldots, \phi_m\}$. It follows that the action of L_m is fully determined by m elements of its kernel:

$$L_m(f) = \frac{W_m[\phi_1, \ldots, \phi_m, f]}{W_m[\phi_1, \ldots, \phi_m]}. \tag{3.4}$$

Here $W_m[\phi_1, \ldots, \phi_m]$ is a determinant of the Wronskian matrix:

$$\mathcal{W}_{m\times m} = (\partial_x^{i-1}(\phi_j))_{1\le i,j\le m}, \tag{3.5}$$

which is nonsingular for the linearly independent functions $\phi_1, \ldots, \phi_m$.

In Eq. (3.2) we encounter ψ_i, $i = 1, \ldots, m$ which are elements of the kernel of an adjoint operator $L_m^\dagger = (-1)^m \partial_x^m + (-1)^{m-1}\partial_x^{m-1} u_{m-1} + \cdots - \partial_x u_1 + u_0$ [25,26]:

$$\psi_i = (-1)^{m+i} \frac{W_{m-1}[\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m]}{W_m[\phi_1, \ldots, \phi_m]}, \quad i = 1, \ldots, m, \tag{3.6}$$

with $\hat{\phi}_i$ being omitted from the Wronskian. Eq. (3.3) contains elements of $\{\bar{\phi}_j\}_{j=1}^{m+1}$ and $\{\bar{\psi}_j\}_{j=1}^{m+1}$ in $\operatorname{Ker}(L_{m+1})$ and $\operatorname{Ker}(L_{m+1}^\dagger)$ connected with each other through a version of (3.6):

$$\bar{\psi}_j = (-1)^{m+1+j} \frac{W_{m-1}[\bar{\phi}_1, \ldots, \hat{\bar{\phi}}_j, \ldots, \bar{\phi}_{m+1}]}{W_{m+1}[\bar{\phi}_1, \ldots, \bar{\phi}_m]}, \quad j = 1, \ldots, m+1. \tag{3.7}$$

Let $(\mathcal{W}^{-1}_{m\times m})_{ij}$, $i, j = 0, \dots, m$ be the matrix elements of the inverse of the Wronskian matrix $\mathcal{W}_{m\times m}$. The following relations are then satisfied:

$$\sum_{j=1}^{m} (\mathcal{W}^{-1}_{m\times m})_{ij} \phi_k^{(j-1)} = \delta_{i,k}, \qquad \sum_{k=1}^{m} \phi_k^{(j-1)} (\mathcal{W}^{-1}_{m\times m})_{kl} = \delta_{j,l}. \tag{3.8}$$

By definition it holds that

$$(\mathcal{W}^{-1}_{m\times m})_{ij} = (-1)^{i+j} \frac{\det_{(j,i)} \|\mathcal{W}_{m\times m}\|}{W_m[\phi_1, \dots, \phi_m]}, \tag{3.9}$$

where the determinant on the right-hand side is the minor determinant obtained by extracting the jth row and ith column from the Wronskian matrix $\mathcal{W}_{m\times m}$ given in Eq. (3.5).

The following technical identity, which is valid for an arbitrary function χ, follows directly from (3.5)–(3.9):

$$\sum_{j=1}^{m} (\mathcal{W}^{-1}_{m\times m})_{ij} \chi^{(j-1)} = (-1)^{m+i} \frac{W_m[\phi_1, \dots, \hat{\phi}_i, \dots, \phi_m, \chi]}{W_m[\phi_1, \dots, \phi_m]}, \qquad i = 1, \dots, m. \tag{3.10}$$

Due to the definition (3.6) the column $(\psi_1, \dots, \psi_m)^{\mathrm{T}}$ agrees with the last column in the inverse matrix $\mathcal{W}^{-1}_{m\times m}$. As a consequence of this connection we have a relation

$$\sum_{i=1}^{m} \phi_i^{(k)}(t) \psi_i(t) = \delta_{k,m-1} \quad \text{for } k = 0, 1, \dots, m-1. \tag{3.11}$$

Let us introduce a notation

$$\Phi_j^{(-n)} = \mathcal{L}^{-n+1}(L_m(\bar{\phi}_j)), \qquad \Psi_j^{(-n)} = (\mathcal{L}^{\dagger})^{-n+1}(\bar{\psi}_j),$$
$$j = 1, \dots, m+1, \quad n = 1, \dots, \infty, \tag{3.12}$$

where $\mathcal{L}^{\dagger} = (L_m^{\dagger})^{-1} L_{m+1}^{\dagger}$. For $n = 1$ this reproduces the functions:

$$\Phi_j^{(-1)} = L_m(\bar{\phi}_j), \qquad \Psi_j^{(-1)} = \bar{\psi}_j, \quad j = 1, \dots, m+1, \tag{3.13}$$

which satisfy relations

$$\mathcal{L}(\Phi_j^{(-1)}) = 0, \qquad \mathcal{L}^{\dagger}(\Psi_j^{(-1)}) = 0, \quad j = 1, \dots, m+1. \tag{3.14}$$

From (3.11) we easily derive that

$$\mathrm{Res}_{\partial_x} \mathcal{L}^{-1} = \sum_{j=1}^{m+1} L_m(\bar{\phi}_j) \bar{\psi}_j = \sum_{j=1}^{m+1} \Phi_j^{(-1)} \Psi_j^{(-1)} = 1. \tag{3.15}$$

Define now $F_j = \sum_{n=1}^{\infty} \lambda^{n-1} \Phi_j^{(-n)}$. As pointed out in [7], due to (3.14) the F_j's satisfy

$$\mathcal{L}(F_j) = \lambda F_j. \tag{3.16}$$

The following definition appeared in [16].

Definition 3.1. Define the $(m+1)\times(m+1)$ matrix $M=(M_{ij})_{1\le i,j\le m+1}$ by

$$M_{m+1\,j}=F_j,\qquad M_{ij}=\partial_x^{-1}(\Psi_i F_j),\quad i=1,\dots,m,\quad j=1,\dots,m+1,\tag{3.17}$$

or in the matrix form

$$M(\mathbf{u},\lambda)=\begin{pmatrix}\partial_x^{-1}(\Psi_1F_1) & \cdots & \partial_x^{-1}(\Psi_1F_{m+1})\\ \vdots & \cdots & \vdots\\ \partial_x^{-1}(\Psi_mF_1) & \cdots & \partial_x^{-1}(\Psi_mF_{m+1})\\ F_1 & \cdots & F_{m+1}\end{pmatrix}.\tag{3.18}$$

Due to (3.16) and (3.17) we find

$$\begin{aligned}&\partial_x M_{ij}=\Psi_i M_{m+1j},\qquad i=1,\dots,m,\\ &(\partial_x-\lambda)M_{m+1j}+\sum_{i=1}^{m}\Phi_i M_{ij}=0,\qquad j=1,\dots,m+1.\end{aligned}\tag{3.19}$$

In terms of the matrix Lax operator L from (2.18) the above Eq. (3.19) is nothing but spectral problem $L(M)=0$ (2.19). This shows that the matrix M constructed in terms of the objects belonging to the pseudo-differential calculus can be identified with the positive grade dressing matrix $M(\mathbf{u},\lambda)$ of the extended Riemann–Hilbert problem.

Expanding now the $M(\mathbf{u},\lambda)$ as in

$$M(\mathbf{u},\lambda)=\sum_{i=1}^{\infty}M_i(\mathbf{u})\lambda^i=M_0+M_1\lambda+\cdots,\tag{3.20}$$

we find from the definition (3.17) for the zero-grade component of $M(\mathbf{u},\lambda)$:

$$\begin{aligned}&(M_0)_{m+1j}=\Phi_j^{(-1)},\qquad (M_0)_{ij}=\partial_x^{-1}(\Psi_i\Phi_j^{(-1)})),\\ &i=1,\dots,m,\quad j=1,\dots,m+1,\end{aligned}\tag{3.21}$$

or in the matrix form

$$M_0(\mathbf{u})=\begin{pmatrix}\partial_x^{-1}(\Psi_1\Phi_1^{(-1)}) & \cdots & \partial_x^{-1}(\Psi_1\Phi_{m+1}^{(-1)})\\ \vdots & \cdots & \vdots\\ \partial_x^{-1}(\Psi_m\Phi_1^{(-1)}) & \cdots & \partial_x^{-1}(\Psi_m\Phi_{m+1}^{(-1)})\\ \Phi_1^{(-1)} & \cdots & \Phi_{m+1}^{(-1)}\end{pmatrix}.\tag{3.22}$$

We now propose the following matrix as an inverse of M_0:

$$\begin{aligned}&(M_0)^{-1}_{jm+1}=\Psi_j^{(-1)},\qquad (M_0)^{-1}_{ji}=\partial_x^{-1}(\Phi_i\Psi_j^{(-1)}),\\ &i=1,\dots,m,\quad j=1,\dots,m+1,\end{aligned}\tag{3.23}$$

which in the matrix form is given by

$$M_0^{-1}(\mathbf{u}) = \begin{pmatrix} \partial_x^{-1}(\Phi_1 \Psi_1^{(-1)}) & \cdots & \partial_x^{-1}(\Phi_m \Psi_1^{(-1)}) & \Psi_1^{(-1)} \\ \vdots & \cdots & \cdots & \vdots \\ \partial_x^{-1}(\Psi_1 \Psi_{m+1}^{(-1)}) & \cdots & \partial_x^{-1}(\Psi_m \Psi_{m+1}^{(-1)}) & \Psi_{m+1}^{(-1)} \end{pmatrix}. \tag{3.24}$$

We now provide a proof of this statement. We first notice that $\sum_{j=1}^{m+1}(M_0)_{m+1j}(M_0)^{-1}_{jm+1} = 1$ due to (3.15). Furthermore, $\sum_{j=1}^{m+1}(M_0)_{m+1\,j}(M_0)^{-1}_{ji} = 0$ and $\sum_{j=1}^{m+1}(M_0)_{m+1j}(M_0)^{-1}_{ji} = 0$ as follows from relations $\mathcal{L}^{-1}(\Phi_i) = 0$, $(\mathcal{L}^{\dagger})^{-1}(\Psi_i) = 0$, $i = 1, \ldots, m$.

It remains to show the $m \times m$ identities:

$$\sum_{j=1}^{m+1}(M_0)_{ij}(M_0)^{-1}_{jk} = \sum_{j=1}^{m+1} \partial_x^{-1}(\Psi_i \Phi_j^{(-1)})\partial_x^{-1}(\Phi_k \Psi_j^{(-1)}) = \delta_{ik}. \tag{3.25}$$

Consider first a factor:

$$\begin{aligned} \partial_x^{-1}\left(\Phi_k \Psi_j^{(-1)}\right) &= \partial_x^{-1}(L_{m+1}(\phi_k)\bar{\psi}_j) = (-1)^{m+1+j}\partial_x^{-1} \\ &\quad \times \left(\frac{W[\bar{\phi}_1, \ldots, \bar{\phi}_{m+1}, \phi_k]W[\bar{\phi}_1, \ldots, \hat{\bar{\phi}}_j, \ldots, \bar{\phi}_{m+1}]}{W^2[\bar{\phi}_1, \ldots, \bar{\phi}_{m+1}]}\right) \\ &= (-1)^{m+1+j}\frac{W[\bar{\phi}_1, \ldots, \hat{\bar{\phi}}_j, \ldots, \bar{\phi}_{m+1}, \phi_k]}{W[\bar{\phi}_1, \ldots, \bar{\phi}_{m+1}]}, \end{aligned} \tag{3.26}$$

where use was made of the Jacobi theorem:

$$\partial_x\left(\frac{W_{k-1}(f)}{W_k}\right) = \frac{W_k(f)W_{k-1}}{W_k^2}, \tag{3.27}$$

involving Wronskians:

$$W_k \equiv W_k[g_1, \ldots, g_k], \qquad W_{k-1}(f) \equiv W_k[g_1, \ldots, g_{k-1}, f], \tag{3.28}$$

for some arbitrary functions $g_1, \ldots, g_k$.

Using the same technique we also find that

$$\partial_x^{-1}(\Psi_i \Phi_j^{(-1)})) = \partial_x^{-1}(\psi_i L_m(\bar{\phi}_j) = (-1)^{m+j}\frac{W[\phi_1, \ldots, \hat{\phi}_j, \ldots, \phi_m, \bar{\phi}_j]}{W[\phi_1, \ldots, \phi_m]}. \tag{3.29}$$

Therefore, using relation (3.10) the quantity in (3.25) becomes

$$\begin{aligned}
&\sum_{j=1}^{m+1} \partial_x^{-1}(\Psi_i \Phi_j^{(-1)})\partial_x^{-1}(\Phi_k \Psi_j^{(-1)}) \\
&\quad = \sum_{j=1}^{m+1}\sum_{s=1}^{m}(\mathcal{W}_{m\times m}^{-1})_{is}\bar{\phi}_j^{(s-1)} \sum_{l=1}^{m+1} \phi_k^{(l-1)}(\mathcal{W}_{m+1\times m+1}^{-1})_{jl} \\
&\quad = \sum_{s=1}^{m}\sum_{l=1}^{m+1} \delta_{sl}(\mathcal{W}_{m\times m}^{-1})_{is}\phi_k^{(l-1)} = \delta_{ik}
\end{aligned} \tag{3.30}$$

as required for the proof of M_0^{-1} being an inverse of M_0.

Next, define $G_j(\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1}\Psi_j^{(-n)}$ which is the solution of the conjugated spectral problem $\mathcal{L}^{\dagger}(G_j(\lambda)) = \lambda G_j(\lambda)$.

We now construct the inverse of the M matrix which we denoted by $M^{-1} = (M_{ij}^{-1})_{1\le i,j\le m+1}$. In view of $L(M) = 0$ in (2.19) the matrix elements of M^{-1} must satisfy

$$\begin{aligned}
&\partial_x M_{ji}^{-1} = \Phi_i M_{jm+1}^{-1}, \quad i = 1,\ldots,m, \\
&(\partial_x + \lambda)M_{jm+1}^{-1} + \sum_{i=1}^{m} \Psi_i M_{ji}^{-1} = 0, \quad j = 1,\ldots,m+1,
\end{aligned} \tag{3.31}$$

or in the matrix notation

$$\partial_x M^{-1} + M^{-1}(E_{m+1m+1}^{(1)} - A) = 0. \tag{3.32}$$

This provides a recurrence relation which allows calculation of terms of higher grade in M^{-1} from M_0^{-1}. The result in terms of $G_j(\lambda)$ gives the $(m+1)\times(m+1)$ matrix M^{-1} given by

$$M_{jm+1}^{-1} = G_j, \qquad M_{ji}^{-1} = \partial_x^{-1}(\Phi_i G_j), \quad i = 1,\ldots,m, \quad j = 1,\ldots,m+1. \tag{3.33}$$

Observe, that

$$\sum_{j=1}^{m+1} F_j \partial_x^{-1} G_j = \sum_{j=1}^{m+1}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \lambda^n \Phi_j^{(-n+l-1)}\partial_x^{-1}\Psi_j^{(-l-1)} = \sum_{n=0}^{\infty} \lambda^n \mathcal{L}^{-(n+1)}, \tag{3.34}$$

where $\mathcal{L}^{-n} = \sum_{j=1}^{m+1}\sum_{l=1}^{n} \Phi_j^{(-n+l-1)}\partial_x^{-1}\Psi_j^{(-l)}$ generalizes expression for $\mathcal{L}^{-1}$. The result (3.33) regarding the M^{-1} is further supported by the following identities:

$$\sum_{j=1}^{m+1} F_j G_j = \mathrm{Res}_{\partial_x}\left(\sum_{j=1}^{m+1} F_j \partial_x^{-1} G_j\right) = \mathrm{Res}_{\partial_x}\mathcal{L}^{-1} = 1, \tag{3.35}$$

which follows from (3.34) and

$$\sum_{j=1}^{m+1} F_j \partial_x^{-1}(G_j \Phi_i) = 0, \qquad \sum_{j=1}^{m+1} G_j \partial_x^{-1}(F_j \Psi_i) = 0, \quad i = 1, \ldots, m, \tag{3.36}$$

following from $\mathcal{L}^{-1}(\Phi_i) = 0$ and $\mathcal{L}^{\dagger -1}(\Psi_i) = 0$ valid for $i = 1, \ldots, m$.

Here we illustrate the above construction for $m = 1$ with the AKNS Lax operator $\mathcal{L} = \partial_x - r\partial_x^{-1} q$ defining the spectral problem $\mathcal{L}(\psi) = \lambda\psi$. The self-commuting isospectral flows ($n > 0$): $\partial_n r = B_n(r)$ and $\partial_n q = -B_n^{\dagger}(q)$ with $B_n = (\mathcal{L}^n)_+$ belong to the positive part of the AKNS hierarchy. $\mathcal{L}$ can be described as a ratio of two ordinary monic differential operators as $\mathcal{L} = L_2 L_1^{-1}$, where L_1, L_2 denote monic operators $L_1 = (\partial_x + \varphi_1' + \varphi_2')$ and $L_2 = (\partial_x + \varphi_1')(\partial_x + \varphi_2')$ of, respectively, order 1 and 2. A monic differential operator L_2 is fully characterized by elements of its kernel, $\varphi_1 = \exp(-\varphi_2)$ and $\varphi_2 = \exp(-\varphi_2)\int^x \exp(\varphi_2 - \varphi_1)$. Its inverse L_2^{-1}, is given by $L_2^{-1} = \sum_{\alpha=1}^{2} \varphi_\alpha \partial_x^{-1} \psi_\alpha$, where $\psi_1 = -\exp(\varphi_1)\int^x \exp(\varphi_2 - \varphi_1)$ and $\psi_2 = \exp(\varphi_1)$ are kernel elements of the conjugated operator $L_2^{\dagger} = (-\partial_x + \varphi_2')(-\partial_x + \varphi_1')$, see [26] and references therein. In this notation, $\mathcal{L} = \partial_x + L_2(\exp(-\varphi_1 - \varphi_2))\,\partial_x^{-1} \exp(\varphi_1 + \varphi_2)$ and accordingly

$$q = -\exp(\varphi_1 + \varphi_2), \qquad r = -(\varphi_1'' - \varphi_1'\varphi_2') \exp(-\varphi_1 - \varphi_2). \tag{3.37}$$

Similarly, the inverse of $\mathcal{L}$ is also given as a ratio of differential operators $\mathcal{L}^{-1} = L_1 L_2^{-1} = \sum_{\alpha=1}^{2} L_1(\varphi_\alpha)\partial_x^{-1}\psi_\alpha$. The functions $\Phi_\alpha^{(-1)} \equiv L_1(\varphi_\alpha)$ and $\Psi_\alpha^{(-1)} \equiv \psi_\alpha$ satisfy the same flow equations as r and q with respect to the positive flows of the AKNS hierarchy.

To facilitate comparison with the Section 4.4 we introduce variables R, u, v:

$$R = \varphi_1, \qquad u = \mathrm{e}^{\varphi_1} \int^x \mathrm{e}^{\varphi_2 - \varphi_1}, \qquad v = \varphi_1' \,\mathrm{e}^{-\varphi_2} \tag{3.38}$$

in terms of φ_1 and φ_2 [24].

Based on expressions (3.21)–(3.23) obtained by the pseudo-differential approach we write M_0 and its inverse as

$$M_0 = \begin{pmatrix} \partial_x^{-1}(\Phi \Psi_1^{(-1)}) & \Psi_1^{(-1)} \\ \partial_x^{-1}(\Phi \Psi_2^{(-1)}) & \Psi_2^{(-1)} \end{pmatrix}, \qquad M_0^{-1} = \begin{pmatrix} \partial_x^{-1}(\Psi \Phi_1^{(-1)}) & \partial_x^{-1}(\Psi \Phi_2^{(-1)}) \\ \Phi_1^{(-1)} & \Phi_2^{(-1)} \end{pmatrix}, \tag{3.39}$$

and within the constrained AKNS we obtain

$$\begin{aligned} \Phi_1^{(-1)} &= \varphi_1' \,\mathrm{e}^{-\varphi_2} = -u = \Psi_1^{(-1)} = -\mathrm{e}^{\varphi_1} \int^x \mathrm{e}^{\varphi_2 - \varphi_1}, \\ \Phi_2^{(-1)} &= \varphi_1' \,\mathrm{e}^{-\varphi_2} \int^x \mathrm{e}^{\varphi_2 - \varphi_1} + \mathrm{e}^{-\varphi_1} = \Delta\, \mathrm{e}^{-R} = \Psi_2^{(-1)} = \mathrm{e}^{R}. \end{aligned} \tag{3.40}$$

Recalling, that for the CKP hierarchy [16,32]

$$\Psi = \Phi = -q = r, \tag{3.41}$$

and using (3.37), we obtain

$$\partial_x^{-1}(\Psi\Phi_1^{(-1)}) = \mathrm{e}^R, \qquad \partial_x^{-1}(\Psi\Phi_2^{(-1)}) = u. \tag{3.42}$$

In a similar way, we find

$$\partial_x^{-1}(\Phi\Psi_1^{(-1)}) = \mathrm{e}^R, \qquad \partial_x^{-1}(\Phi\Psi_2^{(-1)}) = -u. \tag{3.43}$$

4. The multidimensional Toda model and the Cecotti–Vafa equations

4.1. The multidimensional Toda model

Introduce a notation:

$$\partial_j \equiv \frac{\partial}{\partial u_j^{(1)}}, \qquad \partial_{-j} \equiv \frac{\partial}{\partial u_j^{(-1)}}. \tag{4.1}$$

With this notation the relevant part of the flow equations (2.5)–(2.45) takes a form

$$\begin{aligned} \partial_j \Theta(\mathbf{u},\lambda) &= -(\Theta E_{jj}^{(1)}\Theta^{-1})_- \Theta(\mathbf{u},\lambda), \\ \partial_j M(\mathbf{u},\lambda) &= (\Theta E_{jj}^{(1)}\Theta^{-1})_+ M(\mathbf{u},\lambda), \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \partial_{-j} \Theta(\mathbf{u},\lambda) &= (ME_{jj}^{(-1)}M^{-1})_- \Theta(\mathbf{u},\lambda), \\ \partial_{-j} M(\mathbf{u},\lambda) &= -(ME_{jj}^{(-1)}M^{-1})_+ M(\mathbf{u},\lambda). \end{aligned} \tag{4.3}$$

The last equation in (4.3) can also be rewritten as

$$\begin{aligned} \partial_{-j} M(\mathbf{u},\lambda) &= -(ME_{jj}^{(-1)}M^{-1} - (ME_{jj}^{(-1)}M^{-1})_-)M \\ &= -ME_{jj}^{(-1)} + M_0 E_{jj}^{(-1)} M_0^{-1} M. \end{aligned} \tag{4.4}$$

Projecting (4.4) on the zero grade and recalling expansion in (3.20) we find

$$\partial_{-j} M_0 = -M_1 E_{jj} + M_0 E_{jj} M_0^{-1} M_1, \tag{4.5}$$

which can be cast in the following form:

$$M_0^{-1}\partial_{-j} M_0 = [E_{jj}, M_0^{-1} M_1]. \tag{4.6}$$

Similarly, by projecting the second equation in (4.2) on grades zero and one, we find

$$\partial_j M_0 = [\theta^{(-1)}, E_{jj}]M_0, \quad j = 1, \ldots, m+1, \tag{4.7}$$

$$\partial_i M_1 = E_{jj} M_0 + [\theta^{(-1)}, E_{jj}]M_1. \tag{4.8}$$

Using (4.5) and (4.7) we can cast the flow equations (4.2) and (4.3) in a way which reveals a symmetry between the negative and positive flows and the dressing matrices of the positive and negative gradations. Replacing $M(\mathbf{u},\lambda)$ by $M_0^{-1}M(\mathbf{u},\lambda)$ in (4.2) and (4.3) we find

$$\partial_{-j}\Theta(\mathbf{u},\lambda) = (M_0 E_{jj}^{(-1)} M_0^{-1})\Theta(\mathbf{u},\lambda),$$
$$\partial_j(M_0^{-1}M(\mathbf{u},\lambda)) = (M_0^{-1}E_{jj}^{(1)}M_0)(M_0^{-1}M(\mathbf{u},\lambda)), \tag{4.9}$$

and

$$\partial_j\Theta(\mathbf{u},\lambda) = -(\Theta E_{jj}^{(1)}\Theta^{-1})_-\Theta(\mathbf{u},\lambda),$$
$$\partial_{-j}(M_0^{-1}M(\mathbf{u},\lambda)) = -(M_0^{-1}ME_{jj}^{(-1)}(M_0^{-1}M)^{-1})_{>0}(M_0^{-1}M(\mathbf{u},\lambda)). \tag{4.10}$$

Eqs. (4.9) and (4.10) exhibit invariance under simultaneous interchanges: $\partial_j \leftrightarrow \partial_{-j}$, $\Theta(\mathbf{u},\lambda) \leftrightarrow M_0^{-1}M(\mathbf{u},\lambda)$, $M_0 \leftrightarrow M_0^{-1}$ and of the positive and negative grades. This type of symmetry will be responsible for appearance of the complex like structure among the Cecotti–Vafa equations we will derive later in this section.

Applying now ∂_i on Eq. (4.6) and using relations (4.7) and (4.8) we obtain

$$\partial_i(M_0^{-1}\partial_{-j}M_0) = [E_{jj}, -M_0^{-1}[\theta^{(-1)}, E_{ii}]M_1 + M_0^{-1}(E_{ii}M_0 + [\theta^{(-1)}, E_{ii}]M_1)], \tag{4.11}$$

which after the cancellation of two identical terms with opposite signs results in

$$\partial_i(M_0^{-1}\partial_{-j}M_0) = [E_{jj}, M_0^{-1}E_{ii}M_0], \quad i,j = 1,\ldots,m+1. \tag{4.12}$$

By multiplying both sides of (4.12) by M_0 from the left and M_0^{-1} from the right we obtain an equivalent expression

$$\partial_{-j}(\partial_i M_0 M_0^{-1}) = [M_0 E_{jj} M_0^{-1}, E_{ii}], \quad i,j = 1,\ldots,m+1, \tag{4.13}$$

which can be rewritten as a Toda zero-curvature equation:

$$[\partial_{-j} - M_0 E_{jj}^{(-1)} M_0^{-1}, \partial_i - E_{ii}^{(1)} - (\partial_i M_0)M_0^{-1}] = 0. \tag{4.14}$$

Consider next

$$\partial_{-i}(M_0 E_{jj} M_0^{-1}) = (-M_1 E_{ii} + M_0 E_{ii} M_0^{-1} M_1) E_{jj} M_0^{-1}$$
$$+ M_0 E_{jj}(M_0^{-1} M_1 E_{ii} M_0^{-1} - E_{ii} M_0^{-1} M_1 M_0^{-1}). \tag{4.15}$$

For $i \neq j$ it holds that $E_{ii}E_{jj} = 0$. Using the $i \leftrightarrow j$ symmetry exhibited by two remaining terms on the right-hand side of (4.15) we obtain

$$\partial_{-i}(M_0 E_{jj} M_0^{-1}) = \partial_{-j}(M_0 E_{ii} M_0^{-1}). \tag{4.16}$$

In the same way we also obtain

$$\partial_i(M_0^{-1}E_{jj}M_0) = \partial_j(M_0^{-1}E_{ii}M_0). \tag{4.17}$$

On basis of relations (4.12), (4.16) and (4.17) we recognize that M_0 satisfies the multi-dimensional Toda model [27,28].

4.2. Orthogonal reduction of the $\widehat{\mathrm{gl}}(m+1,\mathbb{C})$-hierarchy

Consider $\hat{\mathcal{G}} = \hat{s}l(m+1)$ with Cartan subalgebra generators $H_a^{(n)}$ and step operators $E_{ij}^{(n)}$ with $n \in \mathbb{Z}$ and $i \neq j$. Next, define the extended automorphism σ, such that

$$\sigma(H_a^{(n)}) = -(-1)^n H_a^{(n)}, \quad a = 1, \ldots, m, \tag{4.18}$$

$$\sigma(E_{ij}^{(n)}) = -(-1)^n E_{ji}^{(n)}, \quad i \neq j = 1, \ldots, m+1. \tag{4.19}$$

The σ automorphism agrees, for $n = 0$, with the well-known automorphism defining the symmetric space $\mathrm{sl}(m+1)/\mathrm{so}(m+1)$ [29]. The combinations

$$H_a^{(2n+1)}, \quad a = 1, \ldots, m, \qquad E_{ij}^{(2n)} - E_{ji}^{(2n)}, \qquad E_{ij}^{(2n+1)} + E_{ji}^{(2n+1)},$$
$$i \neq j = 1, \ldots, m+1, \quad n \in \mathbb{Z}, \tag{4.20}$$

generate the subalgebra of $\widehat{\mathrm{sl}}(m+1)$ invariant under automorphism σ. Let

$$E_\sigma = \tfrac{1}{2}(E^{(1)} + \sigma(E^{(1)})) = \mu_m \cdot H^{(1)}, \tag{4.21}$$

and consider the kernel of $\mathrm{ad}(E_\sigma)$ within the subalgebra of $\widehat{\mathrm{sl}}(m+1)$ invariant under automorphism σ. Such kernel is generated by even combinations from (4.20) within $\hat{s}l(m) \otimes \hat{u}(1)$. The image of $\mathrm{ad}(E_\sigma)$ is generated by the following combinations:

$$E_{m+1i}^{(2n)} - E_{im+1}^{(2n)}, \qquad E_{m+1i}^{(2n+1)} + E_{im+1}^{(2n+1)}, \quad i = 1, \ldots, m, \quad n \in \mathbb{Z}. \tag{4.22}$$

The corresponding reduced potential lies in the zero-grade sub-space spanned by (2.2), i.e.

$$A_\sigma = \sum_{i=1}^{m} \Phi_i (E_{m+1i}^{(0)} - E_{i\,m+1}^{(0)}), \tag{4.23}$$

with the property that

$$A_\sigma = -A_\sigma^{\mathrm{T}}. \tag{4.24}$$

The loop group generalization of the automorphism in (4.18) and (4.19) has the following form [31]:

$$\sigma(X(\lambda)) = ((X(-\lambda))^{\mathrm{T}})^{-1}, \quad X \in G = \widehat{\mathrm{GL}}(m+1). \tag{4.25}$$

One notices that the evolution Eqs. (2.5), (2.6), (2.44) and (2.45) are invariant under the automorphism σ defined in (4.25) for n being an odd integer. As an illustration we consider Eq. (2.5) and find that the flows for the σ transformed Θ matrix become

$$\frac{\partial}{\partial u_j^{(n)}} \sigma(\Theta)(\mathbf{u},\lambda) = (-1)^n (\sigma(\Theta)(\mathbf{u},\lambda) E_{jj}^{(n)} \sigma(\Theta^{-1})(\mathbf{u},\lambda))_- \sigma(\Theta)(\mathbf{u},\lambda). \tag{4.26}$$

Eq. (4.26) agrees with (2.5) for $(-1)^n = -1$ which shows the desired result.

Accordingly, we define the integrable sub-hierarchy by constraining the dressing matrices $\Theta(\mathbf{u},\lambda)$ and $M(\mathbf{u},\lambda)$ to be the fixed points of the loop group automorphism σ (4.25):

$$\Theta^{-1}(\mathbf{u},\lambda) = \Theta^{\mathrm{T}}(\mathbf{u},-\lambda), \tag{4.27}$$

$$M^{-1}(\mathbf{u},\lambda) = M^{\mathrm{T}}(\mathbf{u},-\lambda), \tag{4.28}$$

with $\Theta(\mathbf{u},\lambda)$ and $M(\mathbf{u},\lambda)$ depending only on odd coordinates $\mathbf{u}$: $(u_j^{(n)} = u_j^{(2k+1)})$. From our discussion above it is clear that the odd flows of the reduced sub-hierarchy will preserve the conditions (4.27) and (4.28).

The fixed points of the automorphism σ form a subgroup of $G = \widehat{\mathrm{GL}}(m+1)$, called a twisted loop group of $\mathrm{GL}(m+1)$. In [30,31], the twisted loop group of $\mathrm{GL}(n)$, in the context of n-component KP hierarchy, was used to find solutions of the Darboux–Egoroff system of PDEs.

Note, that from (4.28) we derive the additional orthogonality constraint on M_0:

$$M_0^{\mathrm{T}} = M_0^{-1}, \tag{4.29}$$

For the first term $\theta^{(-1)}$ of expansion of $\Theta = 1 + \theta^{(-1)} + \mathrm{O}(\lambda^{-2})$ the constraint (4.27) implies that $\theta^{(-1)} = \theta^{(-1)\mathrm{T}}$. Hence the reduction based on (4.27) imposes $A^{\mathrm{T}} = -A$ as in (4.24).

Let us now return to the extended Riemann–Hilbert problem (2.43) and restrict it to the reduced model with only odd-flows and with constraints (4.27) and (4.28). By writing the extended Riemann–Hilbert problems (2.43) for both $\Theta^{-1}(\mathbf{u},\lambda)$ and $\Theta^{\mathrm{T}}(\mathbf{u},-\lambda)$ with the constraint (4.27) imposed we obtain

$$\begin{aligned} M^{\mathrm{T}}(\mathbf{u},-\lambda)M(\mathbf{u},\lambda) &= \exp\left(\sum_{j=1}^{m+1}\sum_{k=0}^{\infty} E_{jj}^{(-2k-1)} u_j^{(-2k-1)}\right) g^{\mathrm{T}}(-\lambda)g(\lambda) \\ &\quad \times \exp\left(-\sum_{j=1}^{m+1}\sum_{k=0}^{\infty} E_{jj}^{(-2k-1)} u_j^{(-2k-1)}\right). \end{aligned} \tag{4.30}$$

Due to (4.28) we see that the necessary condition for the g group element is

$$g^{\mathrm{T}}(-\lambda)g(\lambda) = I, \tag{4.31}$$

i.e. $g(\lambda)$ is a fixed point of the automorphism σ. Alternatively, we can derive the reduced sub-hierarchy from the Riemann–Hilbert problem with odd flows defined on the twisted loop group of $\mathrm{GL}(m+1)$.

The tau function for this sub-hierarchy becomes

$$\begin{aligned} \tau(\mathbf{u}) &= \left\langle 0 \left| \exp\left(\sum_{j=1}^{m+1}\sum_{k=0}^{\infty} E_{jj}^{(2k+1)} u_j^{(2k+1)}\right) g(\lambda) \right.\right. \\ &\quad \left.\left. \times \exp\left(-\sum_{j=1}^{m+1}\sum_{k=0}^{\infty} E_{jj}^{(-2k-1)} u_j^{(-2k-1)}\right) \right| 0 \right\rangle, \end{aligned} \tag{4.32}$$

and can be obtained from the original τ-function in (2.59) by putting $u_j^{(\pm 2k)} = 0$ and enforcing on $g(\lambda)$ the condition (4.31). The model is therefore embedded in the CKP hierarchy [3,16].

4.3. Cecotti–Vafa equations

Next, observe that from (4.5) it follows:

$$\sum_{j=1}^{m+1} \partial_{-j} M_0 = 0. \tag{4.33}$$

Using (4.4) and (4.5) we find

$$\partial_{-j}(M_0^{-1}M_1) = [M_0^{-1}M_1, E_{jj}]M_0^{-1}M_1 + [E_{jj}, M_0^{-1}M_2], \tag{4.34}$$

which leads to

$$\sum_{j=1}^{m+1} \partial_{-j}(M_0^{-1}M_1) = 0. \tag{4.35}$$

Moreover, from (4.5) and (4.34) we get for the matrix components $(M_0)_{ik}$ and $(M_0^{-1}M_1)_{ik}$:

$$\partial_{-j}(M_0)_{ik} = (M_0)_{ij}(M_0^{-1}M_1)_{jk}, \quad j \neq k, \tag{4.36}$$

$$\partial_{-j}(M_0^{-1}M_1)_{ik} = (M_0^{-1}M_1)_{ij}(M_0^{-1}M_1)_{jk}, \quad i, j, k \text{ distinct.} \tag{4.37}$$

From

$$\partial_j(M_0^{-1}M_1) = M_0^{-1}E_{jj}M_0, \tag{4.38}$$

we get

$$\partial_j(M_0^{-1}M_1)_{ik} = (M_0^{-1})_{ij}(M_0)_{jk}. \tag{4.39}$$

Similarly, from (4.7) we find

$$\partial_j(M_0)_{ik} = (\theta^{(-1)})_{ij}(M_0)_{jk}, \quad i \neq j. \tag{4.40}$$

For $\theta^{(-1)}$ we find from (4.2),

$$\partial_j\theta^{(-1)} = [E_{jj}, \theta^{(-2)}] + [\theta^{(-1)}, E_{jj}]\theta^{(-1)}, \tag{4.41}$$

from which it follows that

$$\partial_j(\theta^{(-1)})_{ik} = (\theta^{(-1)})_{ij}(\theta^{(-1)})_{jk}, \quad i, j, k \text{ distinct.} \tag{4.42}$$

Also

$$\partial_{-j}\theta^{(-1)} = M_0E_{jj}M_0^{-1}, \tag{4.43}$$

gives

$$\partial_{-j}(\theta^{(-1)})_{ik} = (M_0)_{ij}(M_0^{-1})_{jk}. \tag{4.44}$$

Consider now the reduced case with the orthogonal matrix: $M_0 = (m_{ij})_{1\le i,j\le m+1}$. For simplicity we introduce a notation $M_0^{-1}M_1 = \bar{\mathcal{B}} = (\bar{\beta}_{ij})_{1\le i,j\le m+1}$. Then from (4.6) it follows that

$$M_0^{\mathrm{T}}\partial_{-j}M_0 = [E_{jj}, \bar{\mathcal{B}}] \tag{4.45}$$

for all j such that $1 \le j \le m+1$. Transposing both sides of the matrix relation (4.45) we find that the matrix $M_0^{-1}M_1 = \bar{\mathcal{B}}$ is symmetric ($\bar{\mathcal{B}}^{\mathrm{T}} = \bar{\mathcal{B}}$) for the matrix M_0 satisfying the orthogonality condition (4.29).

We can summarize our results (4.12), (4.16) and (4.17) in the reduced case as

$$\partial_i(M_0^{\mathrm{T}}\partial_{-j}M_0) = [E_{jj}, M_0^{\mathrm{T}}E_{ii}M_0], \tag{4.46}$$

$$\partial_{-i}(M_0E_{jj}M_0^{\mathrm{T}}) = \partial_{-j}(M_0E_{ii}M_0^{\mathrm{T}}), \tag{4.47}$$

$$\partial_i(M_0^{\mathrm{T}}E_{jj}M_0) = \partial_j(M_0^{\mathrm{T}}E_{ii}M_0). \tag{4.48}$$

These equations have been derived by Cecotti and Vafa [14,33]. Imposing the Hermiticity condition $M_0^{\mathrm{T}} = M_0^*$ or $M_0 = M_0^{\dagger}$, we find that Eqs. (4.46)–(4.48) are invariant under complex conjugation $*$ which takes ∂_j to ∂_{-j} and vice versa.

The $\bar{\mathcal{B}}$-matrix elements satisfy

$$\partial_{-j}\bar{\beta}_{ik} = \bar{\beta}_{ij}\bar{\beta}_{jk}, \quad i, j, k \text{ distinct}, \tag{4.49}$$

$$\sum_{j=1}^{m+1}\partial_{-j}\bar{\beta}_{ik} = 0, \tag{4.50}$$

$$\partial_j\bar{\beta}_{ik} = m_{ji}m_{jk}, \tag{4.51}$$

as follows from relations (4.35), (4.37) and (4.39). The first two Eqs. (4.49) and (4.50) for the symmetric $\bar{\mathcal{B}}$ matrix are characteristic for the Egoroff metric.

For the derivatives of matrix elements m_{ij} we find from (4.5), (4.33) and (4.36):

$$\partial_{-j}m_{ik} = m_{ij}\bar{\beta}_{jk}, \quad j \neq k, \tag{4.52}$$

$$\sum_{j=1}^{m+1}\partial_{-j}m_{ik} = 0, \qquad \sum_{j=1}^{m+1}\partial_j m_{ik} = 0. \tag{4.53}$$

These relations couple to additional relations (4.40), (4.43) and (4.44) involving the symmetric matrix $\theta^{(-1)} = (\beta_{ij})_{1\le i,j\le m+1}$:

$$\partial_j m_{ik} = \beta_{ij}m_{jk}, \quad i \neq j, \tag{4.54}$$

$$\partial_j\beta_{ik} = \beta_{ij}\beta_{jk}, \quad i, j, k \text{ distinct}, \tag{4.55}$$

$$\partial_{-j}\beta_{ik} = m_{ij}\, m_{kj} \tag{4.56}$$

with β_{ij} satisfying

$$\sum_{j=1}^{m+1}\partial_j\beta_{ik} = 0. \tag{4.57}$$

We notice a presence in our formalism of two Egoroff systems involving two symmetric matrices $\bar{\mathcal{B}}$ and $\theta^{(-1)}$. The first one in (4.49) and (4.50) is realized in terms of the negative flows $u_j^{(-1)}$, while the second Egoroff system in (4.55) and (4.57) is based on the positive flows $u_j^{(1)}$. Both systems are coupled to each other through the matrix M_0.

The combined system of Eqs. (4.49)–(4.57) exhibits invariance under the simultaneous interchange:

$$\partial_{-j} \leftrightarrow \partial_j, \qquad \bar{\beta}_{ik} \leftrightarrow \beta_{ik}, \qquad m_{ij} \leftrightarrow m_{ji}, \tag{4.58}$$

which maps one Egoroff system into the other. Eqs. (4.49)–(4.57) provide a coordinate form of the Cecotti–Vafa system [14] which appeared in [15,27]. The symmetry (4.58) introduces a complex like structure analogous to complex conjugation on the complex manifold on which the Cecotti–Vafa system was realized in [15].

For completeness we also list the identities

$$\sum_{j=1}^{m+1} \partial_j (M_0^{-1} M_1) = \sum_{j=1}^{m+1} \partial_j \bar{\mathcal{B}} = I, \qquad \sum_{j=1}^{m+1} \partial_{-j} \theta^{(-1)} = I, \tag{4.59}$$

which follow from (4.38) and (4.43).

4.4. *Example: reduction in case of* $\widehat{\mathrm{gl}}(2,\mathbb{C})$*-hierarchy*

The $\widehat{\mathrm{gl}}(2,\mathbb{C})$-hierarchy contains the homogeneous A_1-hierarchy (also known in the literature as the AKNS hierarchy) together with a trivial decoupled scalar field. Accordingly, we only consider the underlying A_1-hierarchy. In [7] the AKNS hierarchy was extended by the "negative" symmetry flows forming the Borel loop algebra. It was shown there how the complex sine-Gordon equation arises as a symmetry flow of the homogeneous A_1-hierarchy. The complex sine-Gordon and the Nonlinear Schrödinger equations appear as lowest negative and second positive flows within the extended hierarchy. Let $\hat{\mathcal{G}} = \widehat{\mathrm{sl}}(2)$ be a loop algebra on which we are given a graded structure $\hat{\mathcal{G}} = \oplus_{n\in\mathbb{Z}} \hat{\mathcal{G}}_n$ with respect to an integral homogeneous gradation defined by the operator $d = \lambda\, \mathrm{d}/\mathrm{d}\lambda$. The algebra $\mathcal{G} = \mathrm{sl}(2,\mathbb{C})$ has a standard basis $E_\alpha = \sigma_+$, $E_{-\alpha} = \sigma_-$ and $H = \sigma_3$. We work within an algebraic approach to the integrable models based on the linear spectral problem $L(M) = 0$ with a matrix Lax operator containing the matrix $A = qE_\alpha + rE_{-\alpha}$ [8]. The second flow of the hierarchy:

$$\partial_2 r = r_{xx} - 2q^2 r, \qquad \partial_2 q = -q_{xx} + 2q^2 r \tag{4.60}$$

gives the familiar non-linear Schrödinger equation.

The flow generated by $E^{(-1)}$ is of special interest and we now provide its zero-curvature formulation. We choose the Gauss decomposition given by the following exponential of terms belonging to zero grade subalgebra $\hat{\mathcal{G}}_0 = \mathrm{sl}(2)$ in order to parameterize M_0 satisfying (4.7)

$$M_0 = \mathrm{e}^{\chi E_{-\alpha}}\, \mathrm{e}^{RH}\, \mathrm{e}^{\psi E_\alpha}, \tag{4.61}$$

and define gauge potentials

$$\mathcal{A}_- = M_0 E^{(-1)} M_0^{-1}, \qquad \mathcal{A}_+ = -\partial_x M_0 M_0^{-1} + E^{(1)}. \tag{4.62}$$

In order to match the number of independent modes in the matrix A we impose two "diagonal" constraints $\mathrm{Tr}(\partial_x M_0 M_0^{-1} H) = \mathrm{Tr}(M_0^{-1}\partial_{-1} M_0 H) = 0$ which effectively eliminate R in terms of ψ and χ. In fact, those constraints reduce the zero grade subspace $\hat{\mathcal{G}}_0 = \mathrm{sl}(2)$ into the coset $\mathrm{sl}(2)/U(1)$. In terms of variables defined in (4.61), these constraints determine the non-local field R as

$$\partial_x R = \frac{v\partial_x u}{\Delta}, \qquad \partial_{-1} R = \frac{u\partial_{-1} v}{\Delta}, \tag{4.63}$$

where

$$u = \psi\, \mathrm{e}^R, \qquad v = \chi\, \mathrm{e}^R, \qquad \Delta = 1 + uv. \tag{4.64}$$

Since, M_0 has been chosen so that after imposition of the constraints (4.63) $\partial_x M_0 M_0^{-1} = -(qE_\alpha + rE_{-\alpha})$ we obtain the following representation for q and r:

$$q = -\frac{\partial_x u}{\Delta}\mathrm{e}^R, \qquad r = -(\partial_x v)\, \mathrm{e}^{-R}, \tag{4.65}$$

and the zero-curvature condition (4.14):

$$[\partial_{-1} + \mathcal{A}_- \partial_x + \mathcal{A}_+] = \partial_{-1}\mathcal{A}_+ - \partial_x\mathcal{A}_- + [\mathcal{A}_-\mathcal{A}_+] = 0 \tag{4.66}$$

leads to the equations of motion:

$$\partial_{-1} q = -\partial_{-1}\left(\frac{\partial_x u}{\Delta}\mathrm{e}^R\right) = u\, \mathrm{e}^R, \tag{4.67}$$

$$\partial_{-1} r = -\partial_{-1}(\partial_x v\, \mathrm{e}^{-R}) = v\Delta\, \mathrm{e}^{-R}. \tag{4.68}$$

Let us now discuss the orthogonal reduction $-q = r = \Phi$ in expression (4.23). This corresponds to setting $v = -u$ and $\mathrm{e}^{2R} = \Delta$ as follows from Eqs. (4.63)–(4.65). Eqs. (4.67) or (4.68) become in this limit:

$$\partial_{-1}\partial_x u + \frac{u\partial_x u\partial_{-1} u}{\Delta} = -u\Delta. \tag{4.69}$$

Using the 2×2 representation of the sl(2) algebra together with the constraint $u = -v$ it follows from (4.61) that under the orthogonal reduction the matrix M_0 takes the following form:

$$M_0 = \begin{pmatrix} \mathrm{e}^R & u \\ -u & \mathrm{e}^R \end{pmatrix}, \qquad M_0^{-1} = \begin{pmatrix} \mathrm{e}^R & -u \\ u & \mathrm{e}^R \end{pmatrix}, \tag{4.70}$$

which reproduces formulas derived from the pseudo-differential approach in Eqs. (3.39), (3.42) and (3.43).

The constraint $M_0^\dagger = M_0$ amounts to choosing u as a purely imaginary function:

$$u = \mathrm{i}\sinh\beta. \tag{4.71}$$

In this parameterization $\mathrm{e}^R = \cosh\beta$. Plugging this into Eq. (4.68) we obtain the sinh-Gordon equation:

$$\partial_{-1}\partial_x\beta = -\tfrac{1}{2}\sinh(2\beta) \tag{4.72}$$

for the reduced hierarchy. One can verify that $M_0^{-1}M_1$ becomes now a symmetric matrix in agreement with Eq. (4.38).

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References

[1] S. Novikov, S.V. Manakov, L.P. Pitaevsky, V.E. Zakharov, Consultants Bureau, Contemporary Soviet Mathematics, 1984, 276 pp.
[2] L.D. Faddeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer, Berlin, 1987, 592 pp.
[3] V.G. Drinfeld, V.V. Sokolov, J. Sov. Math. 30 (1984) 1975;
V.G. Drinfeld, V.V. Sokolov, Sov. Math. Dokl. 23 (1985) 457.
[4] M.F. de Groot, T.J. Hollowood, J.L. Miramontes, Commun. Math. Phys. 145 (1992) 57.
[5] N.J. Burroughs, M.F. de Groot, T.J. Hollowood, J.L. Miramontes, Commun. Math. Phys. 153 (1993) 187, hep-th/9109014.
[6] J. Dorfmeister, H. Gradl, J. Szmigielski, Acta Appl. Math. 53 (1998) 1.
[7] H. Aratyn, L.A. Ferreira, J.F. Gomes, A.H. Zimerman, J. Phys. A 33 (2000) L331, nlin.si/0007002.
[8] H. Aratyn, L.A. Ferreira, J.F. Gomes, A.H. Zimerman, J. Math. Phys. 38 (1997) 1559, hep-th/9509096.
[9] B. Konopelchenko, W. Strampp, J. Math. Phys. 33 (1992) 3676.
[10] Y. Cheng, J. Math. Phys. 33 (1992) 3774.
[11] W. Oevel, W. Strampp, Commun. Math. Phys. 157 (1993) 51.
[12] H. Aratyn, L.A. Ferreira, J.F. Gomes, A.H. Zimerman, Nucl. Phys. B 402 (1993) 85.
[13] L.A. Dickey, Lett. Math. Phys. 35 (1995) 229, arXiv:hep-th/9411005.
[14] S. Cecotti, C. Vafa, Nucl. Phys. B 367 (1991) 359.
[15] B. Dubrovin, Commun. Math. Phys. 152 (1993) 539, hep-th/9206037.
[16] H. Aratyn, J. van de Leur, hep-th/0104092.
[17] G. Wilson, The τ-function of the g AKNS equations, in: O. Babelon, et al. (Eds.), Proceedings of the Verdier Memorial Conference on Integrable Systems, Boston, MA, Birkhauser, Basel, Prog. Math. 115 (1993) 131.
[18] M.A. Semenov-Tian-Shansky, Lectures on integrable systems, in: O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach (Eds.), Memory of Jean-Louis Verdier Proceedings, Cimpa School, Nice, France, 10–28 June 1991, World Scientific, Singapore, 1994, 269, 354 pp.
[19] A.P. Fordy, P.P. Kulish, Commun. Math. Phys. 89 (1983) 427.
[20] H. Flaschka, A.C. Newell, T. Ratiu, Physica D 9 (1983) 300.
[21] H. Aratyn, J.F. Gomes, A.H. Zimerman, J. Math. Phys. 36 (1995) 3419, hep-th/9408104.
[22] H. Aratyn, J.F. Gomes, E. Nissimov, S. Pacheva, A.H. Zimerman, in: H. Aratyn, A. Sorin (Eds.), Integrable Hierarchies and Modern Physical Theories, Kluwer Academic Publishers, 2001, p. 243, nlin.si/0012042.
[23] I. Krichever, Physica D 87 (1995) 14.
[24] H. Aratyn, J.F. Gomes, E. Nissimov, S. Pacheva, Appl. Anal. 78 (2001) 233, nlin.si/0004040.
[25] W. Oevel, W. Strampp, J. Math. Phys. 37 (1996) 6213.
[26] H. Aratyn, Lecture Notes in Physics, Vol. 502, Springer, Berlin, 1998;
H. Aratyn, J. Geom. Phys. 3 (1999) 295, solv-int/9805006.
[27] A.V. Razumov, M.V. Saveliev, Theor. Math. Phys. 112 (1997) 999, hep-th/9609031.
[28] L.A. Ferreira, J.F. Gomes, A.V. Razumov, M.V. Saveliev, A.H. Zimerman, Commun. Math. Phys. 203 (1999) 649, math-ph/9807017.
[29] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.

[30] V.G. Kac, Infinite Dimensional Lie Algebras, 3rd Edition, Cambridge University Press, Cambridge, 1990.
[31] J. van de Leur, Int. Math. Res. Notices 11 (2001) 551, nlin.si/0004021.
[32] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, J. Phys. Soc. Jpn. 50 (1981) 3813.
[33] S. Cecotti, C. Vafa, Commun. Math. Phys. 157 (1993) 139, hep-th/9209085.